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Quantum Theory of Self-Induced Transparency

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It is known that classical electromagnetic radiation at a frequency in resonance with energy splittings of atoms in a dielectric medium can be described using the classical sine-Gordon theory. In this paper we quantize the electromagnetic field and compute some quantum effects by using known results from the sine-Gordon quantum field theory. In particular, we compute the intensity of spontaneously emitted radition using the thermodynamic Bethe ansatz with boundary interactions.

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1. Introduction

The importance of integrable non-linear partial differential equations in classical non-linear optics has been recognized for some time. The most well-known example concerns weakly non-linear dielectric media, where only the first non-linear susceptibility is considered important. In this situation, the envelope of the electric field satisfies the non-linear Schrodinger equation. This was first understood theoretically by Hasegawa and Tappert[1]. The solitons predicted in [1] were observed experimentally in [2]. The occurence of these classical solitons in common optical fibers promises to revolutionize high-speed telecommunications.

Our interest in this subject concerns the possibly interesting quantum effects which arise when the electromagnetic field is quantized. The latter quantization amounts to studying an interacting quantum field theory in the classical variables which satisfy the non-linear differential equation. These quantum integrable models have been studied extensively, and many of their properties have already been exactly computed. A priori, one expects quantum effects to be small for macroscopically large objects such as solitons[1]. Nevertheless, using known exact results from the quantum non-linear Schrodinger theory, quantum effects have been predicted and measured[3][4][5][6].

It is well-known that the non-linear dielectric susceptibilities are enhanced when the radiation is in resonance with the energy splitting of quantum states of the atoms of the sample. Near resonance, one is no longer in the weakly non-linear regime (higher susceptibilities involving higher powers of the electric field become as important) and the physics is no longer well described by the non-linear Schrodinger equation. Remarkably, as was understood and demonstrated experimentally by McCall and Hahn[7], the system is well described classically by another famous integrable equation, the sine-Gordon (SG) equation. The phenomenon is referred to as 'self-induced transparency'.

In this paper, we study the quantum effects which arise when electric fields in resonance with a dielectric medium are quantized by using known exact results for the SG quantum field theory. In particular, we compute the intensity of spontaneously emitted radiation by using the thermodynamic Bethe ansatz with boundary interactions[2]. A different but related model is the Dicke Model, which was solved using Bethe ansatz in [9].

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[1] In fiber optic systems the soliton consists of a cluster of 10^8 or more photons.

[2] The first few sections of this paper previously appeared in [8].

2. Classical Theory

In this section we review the manner in which the classical sine-Gordon equation arises in resonant dielectric media[7].

We consider electromagnetic radiation of frequency ω propagating through a collection of atoms, where the frequency ω is in resonance with an energy splitting of the atomic states. For simplicity, we suppose each atom is a two state system described by the hamiltonian H_0 with the following eigenstates: $H_0|\psi_1\rangle = -\frac{1}{2}\hbar\omega_0|\psi_1\rangle$, $H_0|\psi_2\rangle = \frac{1}{2}\hbar\omega_0|\psi_2\rangle$, such that $\hbar\omega_0$ is the energy difference of the two states. In the presence of radiation, the atomic hamiltonian is

$$H_{atom} = H_0 - \vec{p}\cdot\vec{E}, \tag{2.1}$$

where $\vec{p} = e\sum_i \vec{r}_i$ is the electric dipole moment operator.

We assume the radiation is propagating in the $\widehat{x}$ direction, and $\vec{E} = \widehat{n}E(x,t)$, where $\widehat{n}\cdot\widehat{x} = 0$. The only non-zero matrix elements of the operator $\vec{p}\cdot\vec{E}$ can be parameterized as follows:

$$\langle\psi_2|\vec{p}\cdot\vec{E}|\psi_1\rangle = pE(x,t)e^{-i\alpha}, \tag{2.2}$$

where p and α are constants which depend on the atom in question. (We have assumed spherical symmetry.) It is convenient to introduce the Pauli matrices σ_i, and write the hamiltonian as

$$H_{atom} = -\frac{1}{2}\hbar\omega_0\,\sigma_3 - E(x,t)(p_1\sigma_1 + p_2\sigma_2), \tag{2.3}$$

where $p_1 = p\cos\alpha,\ p_2 = p\sin\alpha$.

The dynamics of the system is determined by Maxwell's equations,

$$\left(\partial_x^2 - \frac{1}{\overline{c}^2}\partial_t^2\right)E(x,t) = \frac{4\pi}{c^2}\,\partial_t^2 P(x,t), \tag{2.4}$$

where $\overline{c}^2 = c^2/\epsilon_0$, ϵ_0 is the ambient dielectric constant, and $\vec{P} = \widehat{n}P(x,t)$ is the dipole moment per unit volume. The latter polarization can be expressed in terms of the expectations of the Pauli spin matrices $\langle\sigma_i\rangle = \langle\psi|\sigma_i|\psi\rangle$, where $|\psi\rangle$ is the atomic wavefunction. Namely,

$$\vec{P} = \widehat{n}n\left(p_1\langle\sigma_1\rangle + p_2\langle\sigma_2\rangle\right), \tag{2.5}$$

where n is the number of atoms per unit volume[3]. Thus, in addition to the Maxwell equation (2.4), one has dynamical equations for the polarization $P(x,t)$ which are determined by Schrodinger's equation for the atom:

$$i\hbar\,\partial_t\langle\sigma_i\rangle = \langle\psi|\,[\sigma_i, H_{atom}]\,|\psi\rangle. \tag{2.6}$$

The latter can be expressed as

$$\partial_t\langle\sigma_i\rangle = \sum_{j,k}\varepsilon_{ijk}V_j\ \langle\sigma_k\rangle, \tag{2.7}$$

where ε is the completely antisymmetric tensor with $\varepsilon_{123}=1$, and

$$V_1 = \frac{2E(x,t)}{\hbar}p_1, \qquad V_2 = \frac{2E(x,t)}{\hbar}p_2, \qquad V_3 = \omega_0. \tag{2.8}$$

To summarize, the dynamics is determined from the coupled equations of motion (2.4) and (2.7), wherein the atoms are treated quantum mechanically and the radiation is classical.

Let

$$E(x,t) = \mathcal{E}(x,t)\cos(\omega t - kx) \tag{2.9}$$

where $\omega/k = \overline{c}$ and $\mathcal{E}(x,t)$ is the envelope of the electric field. We will assume the envelope is slowly varying in comparison to the harmonic oscillations: $\partial_t\mathcal{E} \ll \omega\mathcal{E},\ \partial_x\mathcal{E} \ll k\mathcal{E}$. In this approximation, one finds

$$\left(\partial_x^2 - \frac{1}{\overline{c}^2}\partial_t^2\right)E(x,t) \approx \frac{2\omega}{\overline{c}}\left[\left(\partial_x + \frac{1}{\overline{c}}\partial_t\right)\mathcal{E}(x,t)\right]\sin(\omega t - kx). \tag{2.10}$$

Let us define $\langle\sigma_\parallel\rangle, \langle\sigma_\perp\rangle$ as follows

$$\begin{aligned}\langle\sigma_\parallel\rangle &= \langle\sigma_1\rangle\cos(\omega t - kx + \alpha) + \langle\sigma_2\rangle\sin(\omega t - kx + \alpha)\\ \langle\sigma_\perp\rangle &= -\langle\sigma_1\rangle\sin(\omega t - kx + \alpha) + \langle\sigma_2\rangle\cos(\omega t - kx + \alpha).\end{aligned} \tag{2.11}$$

We make the further approximation that $\cos 2(\omega t - kx + \alpha)$ terms in the equations of motion for $\partial_t\langle\sigma_i\rangle$ can be dropped in comparison to $\cos(\omega t - kx + \alpha)$ (and similarly for the

[3] To be more precise, $\langle\sigma_i\rangle$ here represents average over many atoms in a small volume, and is thus a continuous field depending on x, t.

sine terms)[4]. (This amounts to replacing $\sin^2(\omega t - kx + \alpha), \cos^2(\omega t - kx + \alpha)$ by 1/2). One then finds

$$\partial_t \langle \sigma_\parallel \rangle = (\omega - \omega_0)\, \langle \sigma_\perp \rangle \tag{2.12a}$$

$$\partial_t \langle \sigma_\perp \rangle = -\frac{\mathcal{E}(x,t)}{\hbar} p\, \langle \sigma_3 \rangle + (\omega_0 - \omega) \langle \sigma_\parallel \rangle \tag{2.12b}$$

$$\partial_t \langle \sigma_3 \rangle = \frac{\mathcal{E}(x,t)}{\hbar} p\, \langle \sigma_\perp \rangle. \tag{2.12c}$$

Finally, equations (2.4) and (2.10), upon making an approximation analagous to the slowly varying envelope on the RHS of (2.4), lead to

$$\left[(\partial_x + \frac{1}{\overline{c}} \partial_t) \mathcal{E}(x,t) \right] \sin(\omega t - kx) = \frac{2\pi}{c^2} \overline{c} n \omega p \left(\sin(\omega t - kx) \langle \sigma_\perp \rangle - \cos(\omega t - kx) \langle \sigma_\parallel \rangle \right). \tag{2.13}$$

In order to solve these equations, note that $\partial_t (\sum_i \langle \sigma_i \rangle \langle \sigma_i \rangle) = 0$. Thus, if the atoms start out in their ground state $|\psi_1\rangle$, with $\langle \sigma_3 \rangle = 1$, one has $\sum_i \langle \sigma_i \rangle^2 = 1$ for all time. On resonance, when $\omega = \omega_0$, eq. (2.12a) implies $\langle \sigma_\parallel \rangle = 0$ for all time. The constraint $\sum_i \langle \sigma_i \rangle^2 = 1$ can be imposed with the following parameterization:

$$\langle \sigma_\perp \rangle = \sin(\beta_{cl} \phi(x,t)), \qquad \langle \sigma_3 \rangle = \cos(\beta_{cl} \phi(x,t)). \tag{2.14}$$

The parameter β_{cl} is arbitrary at this stage, but will be fixed in the next section. Equations (2.12b, c) now imply

$$\partial_t \phi = -\frac{p}{\beta_{cl} \hbar}\ \mathcal{E}(x,t). \tag{2.15}$$

Inserting (2.15) into (2.13) and defining

$$x' = 2x - \overline{c} t, \tag{2.16}$$

one obtains the SG equation:

$$\left(\partial_t^2 - \overline{c}^2 \partial_{x'}^2 \right) \phi = -\frac{\mu^2}{\beta_{cl}} \sin(\beta_{cl} \phi), \tag{2.17}$$

where

$$\mu^2 = \frac{2\pi n p^2 \omega}{\hbar \epsilon_0}. \tag{2.18}$$

[4] It can be shown in perturbation theory that this is a good approximation at or near resonance.

3. The Quantum Spectrum

In this section we proceed to quantize the electromagnetic field. In order to do this, the SG field ϕ must be properly normalized such that the energy of soliton solutions corresponds to the physical energy; this amounts to properly fixing the constant β_{cl}.

The action which gives the classical SG equation of motion is

$$S_{SG} = \frac{1}{\overline{c}} \int dx' dt \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{\overline{c}^2}{2}(\partial_{x'}\phi)^2 + \frac{\mu^2}{\beta_{cl}^2} \cos(\beta_{cl}\phi) \right). \tag{3.1}$$

On the other hand, the properly normalized Maxwell action is

$$S_{\text{Maxwell}} = \frac{1}{c^2} \int d^3x dt \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + ... \right) \quad = \frac{1}{c^2} \int d^3x dt \ \frac{1}{2}(\partial_t A)^2 + ... \tag{3.2}$$

where the vector potential is $\vec{A} = \widehat{n} A$ and as usual $\vec{E} = -\frac{1}{c}\partial_t \vec{A}$. (One can show that A_0 can be set to zero.) One should be able to show directly that the complete Maxwell action, upon making the appropriate approximations of the last section, reduces to the sine-Gordon action, since we have already shown this to be the case at the level of the equations of motion. Assuming this to be the case, in order to properly normalize the field ϕ, one needs only compare the kinetic terms in (3.1) and (3.2). The dimensional reduction is made by assuming simply that A is independent of y, z and $\int dydz = \mathcal{A}$, where $\mathcal{A}$ is an effective cross-sectional area perpendicular to the direction of propagation (as in the cross-sectional area of a fiber). From (2.15) one has

$$S_{\text{Maxwell}} = \mathcal{A} \left(\frac{\beta_{cl}\hbar}{p} \right)^2 \int dx dt \ \frac{1}{2}(\partial_t \phi)^2 \cos^2(\omega t - kx) + ... \tag{3.3}$$

In the above equation, we make an approximation analagous to the one made in arriving at (2.12) and replace $\cos^2(\omega t - kx)$ by 1/2. Noting that $dxdt = dx'dt/2$, upon comparing with (3.1), one fixes $\beta_{cl}^2 = 4p^2/(\mathcal{A}\hbar^2\overline{c})$.

Finally, as is conventionally done in the quantum SG literature, we rescale $\phi \to \sqrt{\hbar}\phi$, so that $S_{SG}/\hbar$ takes the form (3.1)with β_{cl} replaced by β, where

$$\beta^2 = \hbar \beta_{cl}^2 = \frac{4p^2\sqrt{\epsilon_0}}{\mathcal{A}\hbar c}. \tag{3.4}$$

The constant β is the conventional dimensionless coupling constant in the quantum SG theory, and is allowed to be in the range $0 \leq \beta^2 \leq 8\pi$, where $\beta^2 = 8\pi$ corresponds to

a phase transition[10]. The limit where the radiation is classical but the atom is still quantum mechanical corresponds to the limit $\beta \to 0$.

The classical soliton solutions to the SG equation are characterized by a topological charge $T = \pm 1$, where

$$T = \frac{\beta}{2\pi}(\phi(x' = \infty) - \phi(x' = -\infty)).$$

Solitons of either charge correspond to solutions where at fixed x the atoms in the far past are in their ground state, and are all in their excited state at some intermediate time: $\langle \sigma_3 \rangle_{t=\pm\infty} = 1$. These classical solitons have been observed experimentally in [7]. What distinguishes solitons ($T = 1$) from antisolitons ($T = -1$) is the sign of the envelope of the electric field. From the known classical soliton solutions and (2.15) one finds for $T = \pm 1$,

$$\mathcal{E}(x', t) = \pm \frac{2\hbar\mu}{p} \sqrt{\frac{\overline{c} + v}{\overline{c} - v}} \left(\cosh \left(\frac{\mu(x' - vt)}{\sqrt{\overline{c}^2 - v^2}} \right) \right)^{-1} . \tag{3.5}$$

Thus the electric fields for the soliton versus the antisoliton are out of phase by π.

The particle spectrum of the quantum field theory is known[5] to consist of soliton and anti-soliton of mass m_s, and a spectrum of breathers of mass

$$m_n = 2m_s \, \sin \frac{n\gamma}{16}, \qquad n = 1, 2, ... < \frac{8\pi}{\gamma}, \tag{3.6}$$

where

$$\gamma = \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}}. \tag{3.7}$$

The n-th breather disappears from the spectrum when $8\pi/\gamma = n$; at this threshold the mass of this breather is twice the soliton mass. The n-th breather can thus be considered as a bound state of n soliton/antisoliton pairs. The $n = 1$ breather is known to correspond to the particle associated with the sine-Gordon field ϕ itself.

From (3.2)(3.3), one sees that the sine-Gordon scalar field is essentially the photon vector potential, thus the $n = 1$ breather may be thought of as a 'envelope photon', which is massive due to the dielectric properties of the medium. Of course, this photon has little to do with the real free photons of energy $\hbar\omega$. The n-th breather state can then be viewed as an n-envelope-photon bound state. Under this heuristic assignment of photon number to the spectrum, the soliton has photon number 1/2. In specific physical situations, β is

[5] The literature on the quantum SG theory is extensive. See for example [11], and references therein.

very small (see below), thus $m_1 << m_s$, which indicates the strong binding energy of a pair of solitons to form an envelope photon.

The soliton mass is directly related to the parameter μ in (2.18). In the quantum theory, short distance singularities are removed by suitably normal ordering the $\cos(\beta\phi)$ potential[10]:

$$\frac{\mu^2}{\beta^2}\cos(\beta\phi) \to \lambda : \cos(\beta\phi) :$$

The anomalous scaling dimension of the operator $:\cos(\beta\phi):$ is $\beta^2/4\pi$, so that λ has mass dimension $2(1-\frac{\beta^2}{8\pi})$. Therefore,

$$m_s \propto (\mu)^{1/(1-\frac{\beta^2}{8\pi})}. \tag{3.8}$$

This allows us to obtain quantum corrections to the frequency and density dependence of the soliton mass. Since $\mu \propto \sqrt{\omega n}$, one finds

$$m_s \propto \sqrt{\omega n}\left(1 + \frac{\beta^2}{16\pi}\log(\omega n) + O(\beta^4)\right). \tag{3.9}$$

The exact relation between m_s and μ is known[12]:

$$m_s = c(\beta) m^{\gamma/\beta^2}\Lambda^{-\gamma/8\pi}, \qquad m = \frac{\mu\hbar}{c^2}, \tag{3.10}$$

where Λ is an ultraviolet cutoff, and

$$c(\beta) = \frac{2\Gamma(\gamma/16\pi)}{\sqrt{\pi}\Gamma(\gamma/2\beta^2)}\left[\frac{\pi}{2\beta^2}\frac{\Gamma(1-\frac{\beta^2}{8\pi})}{\Gamma(\gamma^2/8\pi\beta^2)}\right]^{\gamma/2\beta^2}. \tag{3.11}$$

As $\beta \to 0$, one obtains the well known classical expression,

$$m_s = \frac{8m}{\beta^2}, \qquad (\beta \to 0). \tag{3.12}$$

From (3.6), one sees that for small β, m is approximately the mass of the lowest breather.

The magnitude of the quantum corrections to classical results is determined by the parameter $\beta^2/8\pi$. In the original experiment of McCall and Hahn, a ruby sample was used with $p = 4.8 \times 10^{-21}$ in cgs units, which corresponds to $\beta^2 = 10^{-23}\sqrt{\epsilon_0}/\mathcal{A}$. The ruby rod had a cross-sectional area of $\mathcal{A} \approx 1cm^2$. Taking the density n to be that of the Cr^{+3} doping atoms in the ruby, then $n \approx 10^{19}/cm^3$. For $\omega \approx 10^{15}s^{-1}$, one finds $m \approx 10^{-37}g$, and $m_s \approx 10^{-13}g$. This large hierarchy of mass scales between the soliton and the lowest

breather is due to the smallness of β. Thus the experiment performed by McCall and Hahn is in the extreme quasi-classical limit, with an extensive spectrum of light and heavy breathers and relatively heavy solitons with macroscopic masses. In the quantum theory, the classical soliton observed in for example [7], becomes a fundamental particle, i.e. it is not to be thought of as a coherent state of photons, due to the fact that the soliton carries non-zero conserved topological charge.

Since in the limit $\beta \to 0$, $m_s = 8m/\beta^2$, from (3.4) one can express $\beta^2/8\pi$ in terms of the classical soliton mass:

$$\frac{\beta^2}{8\pi} = 16\sqrt{\epsilon_0}\left(\frac{\hbar\omega}{m_s c^2}\right)\left(\frac{\hbar n \mathcal{A} c}{m_s c^2}\right). \tag{3.13}$$

Thus,

$$\frac{\beta^2}{8\pi} \sim \frac{1}{N_\gamma}\left(\frac{\lambda_c}{L_{\rm atom}}\right), \tag{3.14}$$

where $N_\gamma = \hbar\omega/m_s c^2$ roughly corresponds classically to the number of photons that comprise the soliton, λ_c is the Compton wavelength of the soliton, and $L_{\rm atom} = 1/n\mathcal{A}$ is the inter-atomic spacing. The above equation summarizes where one expects quantum effects to be important: when the soliton is comprised of small numbers of photons, or when the Compton wavelength is large compared to the space between the atoms.

The classical scattering matrix for the solitons has been computed by comparing N-soliton solutions in the far past and far future[13]. The exact quantum S-matrix is also known[11]. Since $\beta^2/8\pi$ is small here, the quantum corrections to classical scattering are most easily determined by incorporating the one-loop corrections to the classical scattering, which amounts to the replacement $\beta^2 \to \gamma$. The necessary formulas can be found in the above papers.

The most interesting aspect of the quantum scattering of solitons is that the so-called reflection amplitude is a purely quantum effect, analagous to barrier penetration [14]. Namely, one considers an in-state consisting of a soliton of momentum p_1 and an antisoliton of momentum p_2 which scatters into an out state where the momenta p_1 and p_2 are interchanged. The $T = +1$ soliton is thus reflected back with momentum equal to that of the incoming $T = -1$ antisoliton. In the semi-classical approximation, this reflection amplitude is

$$S_R(\theta) = \frac{1}{2}\left(e^{16\pi^2 i/\gamma} - 1\right) e^{-\frac{8\pi}{\gamma}|\theta|}\; S(\theta) \tag{3.15}$$

where

$$S(\theta) = \exp\left(\frac{8}{\gamma}\int_0^{\pi} d\eta \log\left[\frac{e^{\theta - i\eta} + 1}{e^{\theta} + e^{-i\eta}}\right]\right), \tag{3.16}$$

and $\theta = \theta_1 - \theta_2$, where $p_{1,2} = M_s \sinh\theta_{1,2}$. Since $1/\gamma \approx 1/\beta^2$ is very large, the oscillatory factor $\exp(16\pi^2 i/\gamma) - 1$ in (3.15) will make the detection of reflection processes difficult, since even in a small range of β^2, this factor averages to zero.

4. Spontaneous Emission

In this section we will consider the spontaneous emission of radiation from a collection of atoms in their excited state. We suppose that all the atoms have been 'pumped' by some means to their excited state at $t = 0$. In the coordinate system (x', t'), defined by (2.16) and $t' = t$, the atoms are along the x' axis at $t = 0$. If $|\psi\rangle$ is the initial state corresponding to all atoms being in their excited state, then

$$\langle\psi|\sigma_3|\psi\rangle = -1 = \cos\beta\phi, \qquad (t = 0). \tag{4.1}$$

Thus, the initial state corresponds to the initial constant field configuration

$$\phi = \phi_0 = \frac{\pi}{\beta}. \qquad (t = 0). \tag{4.2}$$

All atoms initially in the ground state corresponds to $\phi_0 = 0$. The subsequent time development of the system thus amounts to an initial value problem in the SG quantum field theory.

The initial condition (4.2) may be imposed by adding to the SG action the boundary term:

$$S_{\text{boundary}} = g\int dx \cos\left(\beta(\phi - \phi_0)/2\right). \tag{4.3}$$

In the limit $g \to \infty$, this enforces the boundary condition $\phi = \phi_0$. It is known that the boundary interaction (4.3) preserves the integrability of the SG theory[15]. Ghoshal and Zamolodchikov have developed a comprehensive approach to studying integrable quantum field theory with boundary interactions, and in particular have solved the boundary SG theory. The limit $g \to \infty$ is referred to as the fixed boundary condition in [15], whereas $g = 0$ is called the free boundary condition. As shown there, information about the boundary condition is concentrated in a boundary state $|B\rangle$ at $t = 0$, with the general form

$$|B\rangle = \exp\left(\int d\theta K^{ab}(\theta) A_a(-\theta) A_b(\theta)\right)|0\rangle \tag{4.4}$$

where A_a are creation operators of particles of type a with rapidity θ, and

$$K^{ab}(\theta) = R^{b}_{\overline{a}}(i\pi/2 - \theta), \tag{4.5}$$

where $R(\theta)$ are the S-matrices for reflection off the boundary, and $\overline{a}$ is the charge conjugate of a. The rapidity parameterizes energy and momentum as follows,

$$E' = m \cosh\theta, \qquad P' = m \sinh\theta, \tag{4.6}$$

where E', P' are energy and momentum in the coordinate system (x', t'). (We have set $\hbar = \overline{c} = 1$.) In our application, since the boundary is a spacial axis at $t = 0$, $K^{ab}(\theta)$ represents an amplitude for emission of particle pairs at the boundary.

In order to compute the intensity, one should first study a partition function with the relevant boundary conditions. In euclidean space, one may consider the partition functions

$$Z(R) = \langle B'| e^{-HR} |B\rangle, \tag{4.7}$$

where H is the hamiltonian, and $\langle B'|$ is a conjugate boundary state. Taking $|B\rangle$ to be the boundary state for the fixed boundary condition with ϕ_0 given in (4.2), and $\langle B'|$ to be the boundary state for the free boundary condition provides information on the time evolution of a state where the atoms are all initially excited, and after a time interval R the final state has no constraints on ϕ.

To simplify the initial discussion, let us assume there is only one type of particle in the spectrum, with bulk two-particle S-matrix $S(\theta)$, and boundary S-matrices $K(\theta)$ and $K'(\theta)$ for the boundary states $|B\rangle$ and $|B'\rangle$ respectively. The function $Z(R)$ has the interpretation as a partition function on a strip of width R with boundary conditions $|B\rangle$ at both ends. As shown in [16], if one uses the expression (4.4) for $|B\rangle$, then one encounters divergences in $Z(R)$ due to the fact that the formula (4.4) is only valid in infinite volume. We thus regulate these divergences by taking the boundary at $t = 0$ to have a large but finite length L. (L corresponds to the length of the sample.) Then $Z(R)$ is characterized by the thermodynamic Bethe ansatz (TBA)[16]. These TBA equations generalize the bulk equations with no boundaries derived in [17]. One introduces a density of pairs of particles per unit length $\rho(\theta)$ and similarly a density of holes $\rho^h(\theta)$. Due to the finite volume L, the momenta are quantized in a way that involves the bulk S-matrix:

$$\rho(\theta) + \rho^h(\theta) = \frac{m}{2\pi} \cosh\theta + \frac{1}{2\pi} \left(\Phi * \rho\right)(\theta), \tag{4.8}$$

where

$$\Phi(\theta) = -i\partial_\theta \log S(\theta), \tag{4.9}$$

and $*$ denotes convolution,

$$\left(\Phi * \rho\right)(\theta) = \int_{-\infty}^{\infty} d\theta' \Phi(\theta - \theta')\rho(\theta'). \tag{4.10}$$

Defining

$$\frac{\rho^h}{\rho} = e^{\varepsilon}, \tag{4.11}$$

then ε satisfies the integral equation

$$\varepsilon(\theta) = 2mR\cosh\theta - \log(K(K')^*) - \frac{1}{2\pi}\Phi * \log\left(1 + e^{-\varepsilon}\right). \tag{4.12}$$

The partition function is then given by the formula

$$\log Z(R) = \frac{mL}{4\pi}\int_{-\infty}^{\infty} d\theta \cosh\theta \log\left(1 + e^{-\varepsilon(\theta)}\right). \tag{4.13}$$

One can now compute the intensity, i.e. the energy flux per unit time of the emitted radiation. This radiation will consist of waves traveling to the right and left. Above, the SG theory describes the envelope for the right moving wave (2.9), therefore we compute the intensity in the right-moving radiation. The TBA analysis in [16] assumes periodic boundary conditions in the spacial direction (cylindrical geometry), however we consider the intensity computed below to be a good model of the measureable intensity at one end of the sample.

In the slowly varying envelope approximation, the Pointing vector is

$$\vec{S} = \frac{c\sqrt{\varepsilon_0}}{4\pi}\mathcal{E}^2(x,t)\cos^2(\omega t - kx)\widehat{x}. \tag{4.14}$$

Averaging over periods of the $\cos\omega t$ oscillations, one has that

$$I = \frac{1}{2}I_{\mathrm{env}}, \tag{4.15}$$

where I_{env} is the intensity of the envelope. I_{env} can be computed from the particle picture of the above TBA. Let (E, P, v), (E', P', v') be the energy, momentum and velocity of a single particle in the (x, t) and (x', t') coordinate systems respectively. One has

$$P = 2P', \qquad E = E' - \overline{c}P', \qquad v = (v' + \overline{c})/2. \tag{4.16}$$

If E', P' are given as in (4.6), then $E = me^{-\theta}$, and $v = (1+\tanh\theta)/2$.

In the coordinate system (x', t'), pairs of particles are emitted with rapidity θ and $-\theta$. For both particles in the pair, $v(\theta) > 0$, thus

$$\begin{aligned} I_{\text{env}} &= \int_0^\infty d\theta\rho(\theta)\left(E(\theta)v(\theta) + E(-\theta)v(-\theta)\right) \\ &= \frac{m\bar{c}^3}{2}\int_{-\infty}^{\infty} d\theta\ \rho(\theta)\frac{1}{\cosh\theta}. \end{aligned} \tag{4.17}$$

(We have reinstated $\bar{c}, \hbar$; ρ has units of $(\text{length})^{-1}$ and satisfies the equation (4.8) with $m \to m\bar{c}/\hbar$.)

In the SG case, for general β, the above TBA equations are very complicated due to the non-diagonal pieces (reflection pieces) of the soliton S-matrix. To simplify the situation, we recall that $\beta^2/8\pi$ is very small in physical applications, and is thus well approximated by

$$\frac{\beta^2}{8\pi} = \frac{1}{N+1}, \tag{4.18}$$

for some integer N. Namely, N is taken as the integer which best approximates (4.18). At the points (4.18), the reflection amplitudes vanish and the scattering becomes diagonal. As explained above, for the problem of spontaneous emission, one should take the boundary state $\langle B'|$ to be for the free boundary condition. This leads to the problem that the solitons, though they scatter diagonally off the fixed boundary condition, scatter off-diagonally from the free boundary condition, and it is not known how to formulate TBA equations in this situation. Instead we solve a different problem corresponding to an initial state wherein all the atoms are in their excited state, and a final state where all the atoms are in their ground state. Both of these are fixed boundary conditions with diagonal soliton scattering off the boundary.

The TBA equations become the following. Introduce densities of particles and holes per unit length for each type of particle $\rho_a(\theta), \rho_a^h(\theta)$ where a runs over breather and soliton indices: $a \in \{1, 2, ..., N-1, +, -\}$, where $\pm$ refers to the soliton and anti-soliton. Introducing

$$\frac{\rho_a^h}{\rho_a} = e^{\varepsilon_a}, \tag{4.19}$$

one has

$$\rho_a + \rho_a^h = \frac{m_a}{2\pi}\cosh\theta + \frac{1}{2\pi}\Phi_{ab} * \rho_b \tag{4.20}$$

$$\varepsilon_a = \nu_a - \frac{1}{2\pi} \sum_b \Phi_{ab} * \log(1 + e^{-\varepsilon_b}) \tag{4.21}$$

$$\log Z(R) = \frac{L}{4\pi} \sum_a \int_{-\infty}^{\infty} d\theta \; m_a \cosh\theta \log(1 + e^{-\varepsilon_a}) \tag{4.22}$$

$$I_{\text{env}} = \sum_a \frac{m_a \overline{c}^3}{2} \int_{-\infty}^{\infty} d\theta \rho_a(\theta) \frac{1}{\cosh\theta}, \tag{4.23}$$

where

$$\nu_a = 2 m_a R \cosh\theta - \log(K_a (K'_a)^*), \tag{4.24}$$

$$\Phi_{ab} = -i\partial_\theta \log S_{ab}(\theta), \tag{4.25}$$

where S_{ab} is the 2-body S-matrix for particles of type a, b and K_a are the boundary S-matrices for fixed boundary condition $\phi_0 = \pi/\beta$, and K'_a are those for the fixed boundary condition $\phi_0 = 0$.

The SG scattering theory at these reflectionless points is equivalent to the minimal scattering theory associated to the root system of the $\widehat{D}_{N+1}$ affine Lie algebra[18] [19]. There is a remarkable universal form of the above TBA equations[20][21]:

$$\varepsilon_a = \nu_a - \frac{1}{2\pi} \sum_b N_{ab} \Phi * \left(\nu_b - \log\left(1 + e^{\varepsilon_b}\right)\right), \tag{4.26}$$

where Φ is the universal kernel:

$$\Phi(\theta) = \frac{N}{\cosh N\theta}, \tag{4.27}$$

and N_{ab} is the incidence matrix for the Dynkin diagram below. Using similar manipulations as in [20][21], one can obtain the universal form of the integral equation for ρ_a, ρ_a^h:

$$\rho_a + \rho_a^h = \nu'_a + \frac{1}{2\pi} \sum_b N_{ab} \Phi * \left(\rho_b^h - \nu'_b\right), \tag{4.28}$$

where

$$\nu'_a = \frac{m_a}{2\pi} \cosh\theta. \tag{4.29}$$

The boundary S-matrices for the fixed boundary conditions can be found in [15][22]. Specializing to the reflectionless points, and using some Gamma function identities, one finds the following[6]. We give below the R reflection amplitudes; the K functions that

[6] The denominator of (5.23) in [15] should be $\Pi(x, \pi/2)\Pi(-x, \pi/2)\Pi(x, -\pi/2)\Pi(-x, -\pi/2)$, and (5.24) should read $\sigma(x, u)\sigma(x, -u) = \cos^2 x \left[\cos(x + \lambda u)\cos(x - \lambda u)\right]^{-1}$.

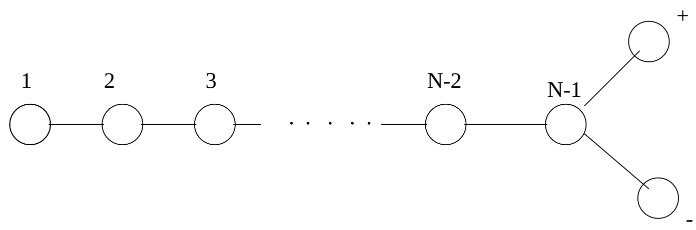

Figure 1. Dynkin diagram for $\widehat{D}_{N+1}$.

appear in the integral equations above are given by $K_a(\theta) = R_a(i\pi/2 - \theta)$. For the solitons, one has

$$R_{\pm}(u) = (-1)^{N-1} \frac{\cos(\xi \pm Nu)}{\cos \xi} \sigma(\xi, u) \prod_{k=0}^{N-1} \frac{\sin(\frac{u}{2} + \frac{\pi k}{4N})}{\sin(\frac{u}{2} - \frac{\pi k}{4N})}, \tag{4.30}$$

where $u = -i\theta$, $\xi = \frac{4\pi}{\beta}\phi_0$, and

$$\sigma(\xi, u) = \prod_{k=0}^{N-1} \frac{\cos\left(\frac{\xi}{2N} + \frac{\pi(1+2k)}{4N}\right) \cos\left(\frac{\xi}{2N} - \frac{\pi(1+2k)}{4N}\right)}{\cos\left(\frac{\xi}{2N} + \frac{\pi(1+2k)}{4N} - u/2\right) \cos\left(\frac{\xi}{2N} - \frac{\pi(1+2k)}{4N} + u/2\right)}. \tag{4.31}$$

For the n-th breather one has

$$R_n(u) = R_0^{(n)}(u) R_1^{(n)}(u), \tag{4.32}$$

$$\begin{aligned} R_0^{(n)}(u) &= (-1)^{n+1} \frac{\cos\left(\frac{u}{2} + \frac{n\pi}{4N}\right) \cos\left(\frac{u}{2} - \frac{\pi}{4} - \frac{n\pi}{4N}\right) \sin\left(\frac{u}{2} + \frac{\pi}{4}\right)}{\cos\left(\frac{u}{2} - \frac{n\pi}{4N}\right) \cos\left(\frac{u}{2} + \frac{\pi}{4} + \frac{n\pi}{4N}\right) \sin\left(\frac{u}{2} - \frac{\pi}{4}\right)} \\ &\times \prod_{k=1}^{n-1} \frac{\sin(u + \frac{k\pi}{2N}) \cos^2\left(\frac{u}{2} - \frac{\pi}{4} - \frac{k\pi}{4N}\right)}{\sin(u - \frac{k\pi}{2N}) \cos^2\left(\frac{u}{2} + \frac{\pi}{4} + \frac{k\pi}{4N}\right)} \end{aligned} \tag{4.33}$$

$$\begin{aligned} R_1^{(2n)}(u) &= \prod_{l=1}^{n} \left(\frac{\sin u - \cos\left(\frac{\xi}{N} - (l - 1/2)\frac{\pi}{N}\right)}{\sin u + \cos\left(\frac{\xi}{N} - (l - 1/2)\frac{\pi}{N}\right)} \right) \left(\frac{\sin u - \cos\left(\frac{\xi}{N} + (l - 1/2)\frac{\pi}{N}\right)}{\sin u + \cos\left(\frac{\xi}{N} + (l - 1/2)\frac{\pi}{N}\right)} \right) \\ &n = 1, 2, \cdots < N/2 \end{aligned} \tag{4.34}$$

$$\begin{aligned} R_1^{(2n-1)}(u) &= \frac{\cos(\xi/N) - \sin u}{\cos(\xi/N) + \sin u} \prod_{l=1}^{n-1} \frac{\sin u - \cos\left(\frac{\xi}{N} - \frac{l\pi}{N}\right)}{\sin u + \cos\left(\frac{\xi}{N} - \frac{l\pi}{N}\right)} \frac{\sin u - \cos\left(\frac{\xi}{N} + \frac{l\pi}{N}\right)}{\sin u + \cos\left(\frac{\xi}{N} + \frac{l\pi}{N}\right)} \\ &n = 1, 2, ..., < \frac{N+1}{2}. \end{aligned} \tag{4.35}$$

5. Concluding Remarks

We have shown how quantized radiation interacting with two-level atoms leads to the quantum sine-Gordon theory in the deep semi-classical regime. As expected, this implies quantum corrections are quite small. We have not considered here the kinds of quantum effects which were studied previously for the non-linear Schrodinger case, which involve taking into account coherent state initial conditions of the radiation in real systems and the squeezing of such states. For the resonant situation considered here, the topological charge of the sine-Gordon solitons presents difficulties for constructing analagous coherent states of photons.

An interesting theoretical question which arises in this work concerns the classical limit, $\beta \to 0$, of the thermodynamic quantities in the SG theory. As suggested above, one can approach this limit by hopping along the refectionless points (4.18). Then the problem becomes that of the N goes to infinity limit of the $\widehat{D}_{N+1}$ TBA equations. Since this corresponds to a certain classical limit of the SG theory, this leads us to believe this limit is meaningful. The interesting limit from a physical point of view is to take $\beta \to 0$, keeping m_s, which tends to $8\mu/\beta_{\rm cl}^2 \overline{c}^2$, fixed. This is reminiscent of the large N studies of the ground state energy for the $SU(N)$ sigma models (principal chiral model), where analytic expressions were obtained in terms of Bessel functions[23]. Another interesting question concerns how sensitive the classical limit is to the way in which β tends to zero. I.e. is the limit different if one does not hop along the reflectionless points? We hope to address these issues in a future publication.

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